Maximum Likelihood Estimation of Pr(X >Y) from Exponential Distribution with Two parameters Under Hybrid Censored Samples

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Abstract

This paper present the estimation of the stress-strength parameter $R = Pr(X > Y)$, when X and Y are two independent exponential random variables with a different scale parameter but having the same location parameter based on hybrid censored samples. The maximum likelihood estimators (MLE) is obtained, the asymptotic distribution of the MLE of R is also obtained, and they have been used to construct the asymptotic confidence interval of R, the Bootstrap-t method is produced and Monte Carlo simulation is presented to evaluate the MLE estimation.

***Key Words and Phrases**:* Hybrid censoring; two-parameter exponential distribution; maximum likelihood estimators; bootstrap confidence interval.

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1. Introduction

It is very important in many applied science such as medicine and engineering…etc to finding the reliability which mean the ability of a system or component to perform its required functions under stated conditions in a specified time. Suppose Y represents the 'stress' which is applied to a certain appliance (every physical component possess an inherent strength) and X represents the 'strength' to sustain the stress, then the stress-strength reliability is denoted by R= Pr$(x > y)$, if X,Y are assumed to be random. In this paper we present an estimation of Pr $(x > y)$ where X and Y are two independent exponential random variables based on hybrid censored samples.

The exponential distribution is one of the most significant, widely used distributions in statistical practice and used extensively in the field of life-testing.

In Type-I Censoring scheme, the experimental time is fixed, but the number of failures is random, where as in Type-II censoring scheme, the experimental time is random but the number of failures is fixed. Epstein (1954) introduced the Type-I hybrid censoring scheme as a mixture of Type-I and Type-II censoring schemes It has been used quite extensively in reliability acceptance, and it is very important in life–testing experiments, it can be described as follows: Suppose a total of n units is placed on a life testing experiment, and the lifetimes of the sample units are independent and identically distributed (i.i.d.) random variables. Let the ordered lifetimes of these items are denoted by $T_{1:n}$; : : : ; $T_{n:n}$ respectively. The test is terminated when r, a pre-chosen number, out of n items are failed (r < n), or when a pre-determined time T on test has been reached. Therefore the Type-I censoring scheme is considered a special case of hybrid censoring when r = n and the Type-II censoring scheme also considered as special case of hybrid censoring when $T \to \infty$. Childs et al(2003) introduced the Type-II hybrid censoring scheme as an alternative to Type-I hybrid censoring scheme, and introduced the exact distribution of the maximum likelihood estimator of the mean of a one parameter exponential distribution based on Type-II hybrid censored samples.

Panerjii and Kundo (2008) introduced the statistical inferences for Weibull parameters when the data are type-II hybrid censored, the maximum likelihood estimators, and the approximate maximum likelihood estimators are presented for estimating the unknown parameters. Asymptotic distributions of the maximum likelihood estimators are used to construct approximate confidence intervals. And they introduced the Bayes estimates, of the unknown parameters. Balakrishnan and Kundu (2013) discussed likelihood inference for the parameters of a two parameter exponential distribution when the data are Type-I hybrid censored. Sanjay, et. Al (2013) presented the Bayes estimators of the parameter and reliability function of inverted exponential distribution under the general entropy loss function for complete data, type I and type II censored samples. Hyun et al (2016) analyzed type-I and type-II hybrid censored data where the lifetimes of items follow two-parameter log-logistic distribution, and they presented the maximum likelihood estimators of unknown parameters and asymptotic confidence intervals, and a simulation study is conducted to evaluate the proposed methods. The estimation of the stress-strength parameter has received more attention in the statistical literature by many authors such as Kundu and Gupta (2005, 2006), Ganguly et al(2012), Gupta et al. (2013), Asgharzadeh et- al(2015), Al-Mutairi et al (2013), Mirjalili et al (2016).

This paper is organized as follows:

In Section 2, the model description is introduced. In section 3, we derive the maximum likelihood estimator (MLE) of R to the exponential distribution with two parameter under hybrid censoring samples. In Section 4, Asymptotic distribution and confidence intervals for R is presented. In Section 5, a Bootstrap-t confidence intervals was presented then in section.6 the simulation study were introduced. Finally in section 7 we conclude on the paper.

2. Model description

Let that the lifetime random variable X has two parameter exponential distribution with the following probability density function PDF:

$$f(x,\beta,\alpha)=\frac{1}{\beta}\exp[-\frac{(x-\alpha)}{\beta}]\ x>0,\beta,\alpha>0 \qquad (1)$$

The distribution function and the reliability function are given by respectively:

$$F(x,\beta,\alpha)=1-\exp[-\frac{(x-\alpha)}{\beta}]x>0,\beta,\alpha>0 \qquad (2)$$

$$R=1-\int_{0}^{x-\alpha} f(t)\,dt=\exp[-\frac{(x-\alpha)}{\beta}] \qquad (3)$$

Where β is the scale parameter and α is the location parameter.

Under the hybrid censoring scheme the observations have one of the two following types:

1: $(x_{1:n}<x_{2:n}<\ldots<x_{h:n})$ *if* $x_{h:n}<T$

2: $(x_{1:n}<x_{2:n}<\ldots<x_{k:n})$ *if* $k<h,\ x_{k:n}<T<x_{k+1:n}$

Where $(x_{1:n}<x_{2:n}<\ldots)$ are the observed ordered failure time to the experimental units.

3. Maximum likelihood estimation of R

Suppose X~exp(α_1, β_1) and Y~ exp (α_2, β_2) are two independent exponential random variables with a different scale parameter but having the same location parameter, then

$$f_1(x)=\frac{1}{\beta_1}\exp[-\frac{(x-\alpha_1)}{\beta_1}]\ and\ \ f_2(y)=\frac{1}{\beta_2}\exp[-\frac{(y-\alpha_2)}{\beta_2}$$

Where $f_1(x)$ and $f_2(y)$ are pdf's of X and Y respectively, then

$$R=\int_{-\infty}^{\infty}\int_{-\infty}^{x}\frac{1}{\beta_1}\exp[-\frac{(x-\alpha_1)}{\beta_2}]\frac{1}{\beta_2}\exp[-\frac{(y-\alpha_2)}{\beta_2}]\ dy\,dx$$

$$R=\frac{\beta_1}{\beta_1+\beta_2}\exp[-\frac{(\alpha_2-\alpha_1)}{\beta_1}]$$

At $\alpha_1 = \alpha_2 = \alpha$ then,

$$R=\frac{\beta_1}{\beta_1+\beta_2} \tag{4}$$

For MLE estimation of R based on hybrid censored data on X and Y we must obtain MLE's of $\alpha, \beta_1 and\, \beta_2$.

Let X=(X_1,X_2,…,X_{r1}) for =($X_{1:n}$,$X_{2:n}$,…,$X_{r1:n}$) is a hybrid censored sample from exp(α, β_1) with censored scheme (r_1,T_1) and Y=(Y_1,Y_2,…,Y_{r2}) for =($Y_{1:n}$,$Y_{2:n}$,…,$Y_{r1:n}$) is a hybrid censored sample from exp(α, β_2) with censored scheme (r_2,T_2).

Therefore the likelihood function in type 1 is given by:

$$L(\alpha,\beta_1)=\frac{n!}{n-h!}(\prod_{i=1}^{h}f(x)(1-F(x_0))^{n-h}) \tag{5}$$

And the likelihood function in Case 2 is given by

$$L(\alpha,\beta_1)=\frac{n!}{n-k!}(\prod_{i=1}^{k}f(x)(1-F(x_0))^{n-k})\ \text{Where}\ 0\le k\le h \tag{6}$$

The equation (5) and (6) can be combined for X and Y random variables; Then the likelihood of X and Y is given by

$$L(\alpha,\beta_1,\beta_2)=\frac{n!}{n-r_1!}(\prod_{i=1}^{r_1}f(x)(1-F(x_0))^{n-r_1})\frac{m!}{m-r_2!}(\prod_{j=1}^{r2}f(y)(1-F(y_0))^{m-r_2}) \tag{7}$$

Where

$$\begin{Bmatrix} r_1 = h \ \ in \ \ type \ \ 1 \\ r_2 = k \ \ in \ \ type \ \ 2 \end{Bmatrix} \ and \ \begin{Bmatrix} x_0 = \min(x_{h1}, T_1) \\ y_0 = \min(x_{h2}, T_2) \end{Bmatrix}$$

$$r_1 = \sum_{i=1}^{r_1} I(x_i \le x_0) \, and \ r_2 = \sum_{j=1}^{r_2} I(y_j \le y_0)$$

Substituting with (1) and (2) in (7)

$$L(data \mid \alpha, \beta_1, \beta_2) = \frac{n!}{n - r_1!} (\prod_{i=1}^{r_1} \frac{1}{\beta_1} \exp[-\frac{(x-\alpha)}{\beta_1}](1 - F(x_0))^{n-r_1}) \\ \frac{m!}{m - r_2!} (\prod_{j=1}^{r2} \frac{1}{\beta_1} \exp[-\frac{(y-\alpha)}{\beta_2}](1 - F(y_0))^{m-r_2}) \qquad (8)$$

From (8) without multiplicative constant

$$L(data \mid \alpha, \beta_1, \beta_2) = \beta_1^{-r_1} \beta_2^{-r_2} \exp \frac{-1}{\beta_1} [\sum_{i=1}^{r_1} (x - \alpha) + (n - r_1) x_0] \\ * \exp \frac{-1}{\beta_2} [\sum_{j=1}^{r_2} (y - \alpha) + (m - r_2) y_0] \qquad (9)$$

The log likelihood function of (9) is

$$\log(data \mid \alpha, \beta_1, \beta_2) = -r_1 \ln \beta_1 - r_2 \ln \beta_2 - \frac{1}{\beta_1} [\sum_{i=1}^{r_1} (x - \alpha) + (n - r_1) x_0] \\ * \frac{1}{\beta_2} [\sum_{j=1}^{r_2} (y - \alpha) + (n - r_2) y_0] \qquad (10)$$

Therefore the MLE of $\alpha, \beta_1 and \, \beta_2$ is given by

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\beta_1} + \frac{m}{\beta_2} = 0 \qquad (11)$$

$\hat{\alpha}$ can be expressed as a constant value

$$\frac{\partial \log L}{\partial \beta_1} = \frac{-r_1}{\beta_1} + \frac{1}{\beta_1^2} [\sum_{i=1}^{r_1} (x - \alpha) + (n - r_1) x_0] = 0 \qquad (12)$$

$$\frac{\partial \log L}{\partial \beta_2} = \frac{-r_2}{\beta_2} + \frac{1}{\beta_2^2}[\sum_{j=1}^{r_2}(y-\alpha)+(m-r_2)y_0] = 0 \qquad (13)$$

From (12) and (13) then

$$\hat{\beta}_1 = \frac{1}{r_1}[\sum_{i=1}^{r_1}(x-\alpha)+(n-r_1)x_0] \qquad (14)$$

$$\hat{\beta}_2 = \frac{1}{r_2}[\sum_{j=1}^{r_2}(y-\alpha)+(m-r_2)y_0] \qquad (15)$$

Substituting the estimation of $\hat{\beta}_1$ *and* $\hat{\beta}_2$ into (4) $\hat{R}$ can be obtained as following:

$$\hat{R} = \frac{\hat{\beta}_1}{\hat{\beta}_1 + \hat{\beta}_2}$$

$$\hat{R} = \frac{r_1^{-1}[\sum_{i=1}^{r_1}(x-\hat{\alpha})+(n-r_1)x_0]}{r_1^{-1}[\sum_{i=1}^{r_1}(x-\hat{\alpha})+(n-r_1)x_0] + r_2^{-1}[\sum_{j=1}^{r_2}(y-\hat{\alpha})+(m-r_2)y_0]} \qquad (16)$$

4. Asymptotic distribution and confidence intervals

In this section we derive the asymptotic distribution of $\hat{\phi} = (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$,then we derive the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, we obtain the asymptotic confidence interval of *R*.

Let the Fisher's information matrix $\phi = (\alpha, \beta_1, \beta_2)$ as $I(\phi) = (I_{ij}(\phi); i, j = 1,2)$ Therefore:

$$I(\phi)=\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = -E\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta_1} & \frac{\partial^2 l}{\partial \alpha \partial \beta_2} \\ \frac{\partial^2 l}{\partial \beta_1 \partial \alpha} & \frac{\partial^2 l}{\partial \beta_1^2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 l}{\partial \beta_2 \partial \alpha} & \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_2^{\,2}} \end{pmatrix} \tag{17}$$

Where

$-I_{11}=-I_{23}=-I_{32}=0$

$$-I_{12}=-I_{21}=\frac{n}{\beta_1^2}$$

$$-I_{13}=-I_{31}=\frac{m}{\beta_2^2}$$

$$-I_{22}=-\frac{r_1}{\beta_1^2}+\frac{2}{\beta_1^3}[\sum_{i=1}^{r_1}(x-\hat{\alpha})+(n-r_1)x_0]$$

$$-I_{33}=-\frac{r_2}{\beta_2^2}+\frac{2}{\beta_2^3}[\sum_{j=1}^{r_2}(y-\hat{\alpha})+(m-r_2)y_0]$$

$$I(\phi)=\begin{pmatrix} 0 & \frac{n}{\beta_1^2} & \frac{m}{\beta_2^2} \\ \frac{n}{\beta_1^2} & -\frac{r_1}{\beta_1^2}+\frac{2}{\beta_1^3}[\sum_{i=1}^{r_1}(x-\hat{\alpha})+(n-r_1)x_0] & 0 \\ \frac{n}{\beta_2^2} & 0 & -\frac{r_1}{\beta_2^2}+\frac{2}{\beta_2^3}[\sum_{j=1}^{r_2}(y-\hat{\alpha})+(m-r_2)y_0] \end{pmatrix}$$

Let $A=I(\phi)^{-1}$ is the asymptotic variance- covariance matrix, where

$$A=\frac{1}{\nabla}\begin{pmatrix} I_{33}I_{22} & -I_{21}I_{33} & -I_{31}I_{22} \\ -I_{12}I_{33} & I_{31}I_{31} & -I_{12}I_{31} \\ -I_{13}I_{22} & I_{13}I_{21} & -I_{12}I_{21} \end{pmatrix}, \ and\ \nabla=-[I_{12}I_{21}I_{33}+I_{13}I_{22}I_{31}] \tag{18}$$

To obtain asymptotic confidence interval for R, we proceed as follows (Rao, 1973):

$$d_1(\beta_1,\beta_2)=\frac{\partial R}{\partial \beta_1}=\frac{\hat{\beta}_2}{(\hat{\beta}_1+\hat{\beta}_2)^2}$$

$$d_2(\beta_1,\beta_2)=\frac{\partial R}{\partial \beta_2}=\frac{-\hat{\beta}_1}{(\hat{\beta}_1+\hat{\beta}_2)^2}$$

This gives

$$\operatorname{var}(\hat{R})=\operatorname{var}(\hat{\beta}_1)d_1^2(\beta_1,\beta_2)+\operatorname{var}(\hat{\beta}_2)d_2^2(\beta_1,\beta_2)$$

$$\therefore \operatorname{var}(\hat{R})=I_{22}\frac{-2\hat{\beta}_2(\hat{\beta}_1+\hat{\beta}_2)}{(\hat{\beta}_1+\hat{\beta}_2)^4}+I_{33}\frac{2\hat{\beta}_1(\hat{\beta}_1+\hat{\beta}_2)}{(\hat{\beta}_1+\hat{\beta}_2)^4} \tag{19}$$

Then the asymptotic $100(1-\lambda)\%$ confidence interval for R would be (L,U).

Where

$$U=\hat{R}+z_{1-\frac{\lambda}{2}}\sqrt{\operatorname{var}(\hat{R})},$$

$$L=\hat{R}-z_{1-\frac{\lambda}{2}}\sqrt{\operatorname{var}(\hat{R})}$$

Where $z_{1-\frac{\lambda}{2}}$ is the $(1-\frac{\lambda}{2})^{th}$ percentile of the standard normal distribution and

$\hat{R}$ is given by equation (16)

5. Bootstrap-t confidence intervals

Efron (1982) introduced bootstrap method as a nonparametric device for estimating standard errors, this method considered as i an automatic algorithm for producing highly accurate confidence limits from a bootstrap distribution bootstrap distribution. The goal of bootstrap confidence interval theory is to calculate dependable confidence limits for a parameter of interest ϑ from the bootstrap distribution of $\hat{\vartheta}$ Thomas (1996). Summarize the way in the following:

1- Given T, r_1,r_2 ,n, m , β_1 *and* β_2 are obtained from (14) and (15)
2- Based on T, r_1,r_2 ,n, m , β_1 *and* β_2 a random sample of size r_1 and r_2 from uniform(0,1) distribution is generated, and order them to get $(x_1^*,...,x_{r1}^*)$ from *(x_1, ...x_{r1})* and $(y_1^*,...,y_{r2}^*)$ from *(y_1, ...y_{r1})*.
3- From step 2 compute the bootstrap estimate of R using (16) say $\hat{R}^*$.
4- The bootstrap estimate of R. Also, compute the statistic

$$T^* = \frac{\hat{R}^* - R}{\sqrt{\mathrm{var}(\hat{R}^*)}}$$

5- Repeat 2 and 3 NBOOT times.
6- Let $\hat{R}_{boot} = F_1^{-1}(x)$, where $F_1(x) = \Pr(T^* \le x)$ be the cdf of T^*, and $\hat{R}_{boot}(x) = \hat{R} + F_1(x)\sqrt{\mathrm{var}(\hat{R})}$ then the 100(1− λ)% confidence interval for R would be (L,U) where $L = \hat{R}_{boot}(\frac{\lambda}{2})$ and $U = \hat{R}_{boot}(1-\frac{\lambda}{2})$

6. Simulation study

In this section, Monte Carlo simulation is presented to illustrate the estimation of R by MLE, asymptotic confidence intervals and Bootstrap-t confidence intervals when n=m=35 and $(\alpha,\beta_1,\beta_2) = (2,1,1)$ We repeat the process for 1000 times and report the average estimates, 95% coverage percentage of asymptotic confidence intervals and Bootstrap-t confidence intervals (for the simulation of Bootstrap-t methods the confidence intervals based on 150 re-sampling). The results are reported in Tables 1.

Table (1): MLE of R, Asymptotic confidence and Bootstrap-t confidence intervals

r_1,T_1	r_2,T_2	MLE	Asymptotic confidence intervals	Bootstrap-t confidence intervals
(15,1)	(15,1)	0.4880	0.4663-0.5097	0.4682-0.5078
	(25,1)	0.4653	0.4442-0.4864	0.4465-0.4841
	(35,1)	0.4372	0.4171-0.4573	0.4265-0.4479
(25,1)	(15,1)	0.4711	0.4516-0.4906	0.4616-0.4806
	(25,1)	0.4315	0.4139-0.4491	0.4227-0.4403
	(35,1)	0.4102	0.3939-0.4265	0.4026-0.4178
(35,2)	(15,2)	0.3974	0.3774-0.4174	0.3911-0.4037
	(25,2)	0.3661	0.3566-0.3756	0.3607-0.3715
	(35,2)	0.3244	0.3162-0.3366	0.3192-0.3296

7. Conclusion

In this paper the maximum likelihood estimation of Pr(X > Y) (as a classical method) when X and Y are independent random variable which Continued two parameter exponential distribution with the same location parameter and different scale parameter is presented, It is assumed that the data are hybrid censored for both X and Y. the Asymptotic distribution and confidence intervals, and Bootstrap-t confidence intervals was introduced.

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