Maximum Likelihood Estimation Of The kumaraswamy Marshal–Olkin Flexible Weibull Extension Distribution under Type-II Censored Samples

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Abstract:

This study presents the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, a newly developed probability model designed to extend existing lifetime distributions. This extension enhances the ability to model various hazard rate patterns with greater flexibility. The study explores the fundamental properties of this distribution, including the cumulative distribution function (CDF), probability density function (PDF), survival function, hazard function, moments, moment generating function, quantile function, and order statistics .To estimate the distribution's parameters, several statistical methods are assessed, with a primary focus on Maximum Likelihood Estimation (MLE). Additionally, alternative estimation techniques such as Maximum Product Spacing (MPS), Least Squares (LS), Weighted Least Squares (WLS), and Percentile Estimation (PE) are considered. A Monte Carlo simulation study is conducted to evaluate the accuracy and efficiency of these

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estimation approaches under both complete data and Type-II censoring conditions .The results indicate that MLE is highly effective for large sample sizes but may introduce bias in smaller samples. In contrast, MPS provides stable parameter estimates under more complex conditions. While LS and WLS are computationally straightforward, their performance diminishes in the presence of censoring. Meanwhile, PE proves beneficial in reliability analysis based on percentile estimation .Overall, the findings highlight the KMO-FWE distribution as a versatile and reliable model, suitable for applications in reliability engineering, survival analysis, and biomedical research.

Key wards :

Kumaraswamy Marshal–Olkin Flexible Weibull Extension, Maximum Likelihood Estimation, Monte Carlo Simulation, Reliability Analysis, Type-II Censoring.

1. Introduction

Analyzing lifetime data and system reliability is essential across various fields, including engineering, medical research, and risk assessment. To model failure times effectively, numerous statistical distributions have been introduced. Among these, the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution has gained recognition for its adaptability in representing different failure rate patterns, such as increasing, decreasing, and bathtub-shaped hazard functions. This flexibility makes it particularly valuable for modeling censored survival data, where complete observations may not always be available due to time constraints .Accurate parameter estimation is a critical aspect of statistical modeling, as it significantly influences the reliability and applicability of a distribution in practical settings. Various estimation approaches have been developed for lifetime distributions, each offering distinct advantages and challenges.

This study focuses on Maximum Likelihood Estimation (MLE), Maximum Product Spacing (MPS), Least Squares (LS), Weighted Least Squares (WLS), and Percentile Estimation (PE) to estimate the parameters of the KMO-FWE distribution under Type I censoring, a widely used framework in survival and reliability studies. weibull distribution is one of the most widely used models in reliability analysis(Nelson, 2005). but various extensions have been introduced to enhance its flexibility. The Flexible Weibull Extension (FWE) distribution, proposed by(Mudholkar, Hutson, & Methods, 1996), allows for more complex hazard rate structures. The Kumaraswamy distribution, introduced by (Kumaraswamy, 1980), provides a flexible alternative to the beta distribution for modeling bounded data. Furthermore, the Marshal-Olkin transformation, proposed by(Marshall & Olkin, 1997)), has been used to add shape parameters to distributions, thereby increasing their adaptability. The Marshall-Olkin Extended Exponential (MOEE) distribution, like the KMO-FWE distribution, introduces an additional shape parameter to improve flexibility, making it more suitable for real-world reliability applications. Studies such as (Ahmad, Almetwally, & Research, 2020) have explored various estimation techniques for MOEE and related distributions, showing that different methods perform variably depending on sample size and censoring level In parameter estimation, MLE remains the most commonly used approach due to its asymptotic efficiency, but it may suffer from convergence issues for complex models.(Fisher, 1922). MPS, introduced by(Cheng & Amin, 1983) , has been proposed as a more stable alternative in certain cases. LS and WLS methods, originally developed for regression analysis, have been adapted for lifetime distributions (Kuhl, Wilson, & Simulation, 2000)). PE, based on quantile properties, has also been explored as a robust and computationally simple method(Arnold, Balakrishnan, & Nagaraja, 2008)).Despite extensive research on flexible lifetime distributions and their estimation techniques, there has been limited investigation into the KMO-FWE distribution with Type I censored samples. This study aims to address this gap by applying multiple estimation methods, evaluating their performance using simulation studies, and validating the results with real-world data. The Kumaraswamy-G family was introduced by (Cordeiro, De Castro, & simulation, 2009) , where its mathematical properties were studied, and special sub-models were presented. The cumulative distribution function (CDF) and probability density function (PDF) are given as follows:

$$F(x; a, b) = 1 - [1 - x^a]^b \qquad a < x < 1 \qquad a, b, > 0$$

$$f(x; a, b) = a\, b\, x^{a-1} (1 - x^a)^{b-1} \qquad o < x < 1 \qquad a, b > 0$$

The Kumaraswamy Marshal-Olkin family of distributions introduced by(Alizadeh et al., 2015a) The cumulative distribution function (CDF) and probability density function (PDF) are given by

$$F(x; a, b, p) = 1 - \left\{1 - \left(\frac{G(x,\zeta)}{1 - \bar{P}\bar{G}(x,\zeta)}\right)^a\right\}^b \qquad (1)$$

$$f(x, a, b, p) = \frac{a\, b\,(1-p)\, g(x,\zeta)\,(G(x,\zeta))^{a-1}}{\left(1 - \bar{p}\bar{G}(x,\zeta)\right)^{a+1}} \left\{1 - \left(\frac{G(x,\zeta)}{1 - \bar{P}\bar{G}(x,\zeta)}\right)^a\right\}^{b-1} \qquad (2)$$

where a , b are shape parameter

Flexible weibull extension distribution introduced by (Bebbington, Lai, Zitikis, & Safety, 2007). Its mathematical properties were explored, and specific sub-models were introduced. The cumulative distribution function (CDF) and probability density function (PDF) are provided as follows:

$$G(x;\alpha,\beta) = 1 - \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\} \qquad x > 0 \qquad (3)$$

$$g(x;\alpha,\beta) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\} \qquad x > 0$$

(4)

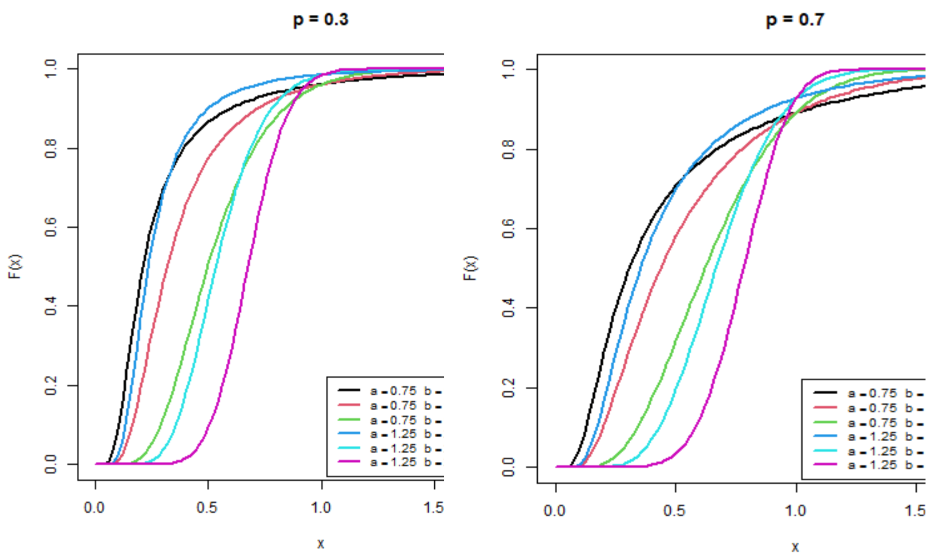
2. Kumaraswamy Marshal – Olkin Flexible Weibull Extension Distribution

In this section, we introduce the Kumaraswamy Marshal–Olkin Flexible Weibull (KMO-FWE) distribution, a newly developed model that extends the Weibull and Flexible Weibull distributions by incorporating the Kumaraswamy and Marshal–Olkin transformations. This new distribution offers enhanced flexibility in modeling lifetime data and reliability analysis by capturing various hazard rate shapes. The cumulative distribution function (CDF) of the KMO-FWE distribution is given from equation (1,3) by :

$$F(x;\Phi) = 1 - \left\{1 - \left[\frac{1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}{1 - \bar{p}\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}\right]^a\right\}^b \qquad x > 0$$

$$a, b, p, \alpha, \beta > 0$$

where a, b, p, α, β and p are the distribution parameters.



Graph (١): Cdf of KMOFWE distribution at different values of *p* and its parameters

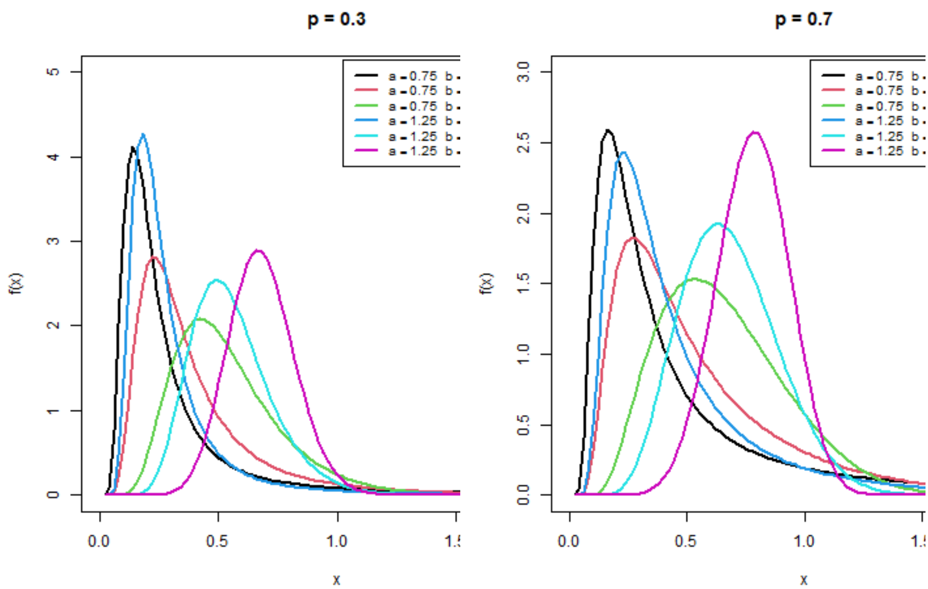
The probability density function (PDF) is given from equation (2,4) by:

$$f(x;\Phi) = a\,b\,(1-p)\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)$$

$$\left\{1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right\}^{a-1} \left(1 - \bar{p}\left[1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right]\right)^{-(\alpha+1)}$$

$$\left\{1 - \left(\frac{1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}{1 - \bar{P}\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}\right)^{\alpha}\right\}^{b-1} \qquad x > 0$$

where $a, b, p, \alpha, \beta > 0$



Graph (٢): Pdf of KMOFWE distribution at different values of *p* and its parameters

This formulation allows the KMO-FWE distribution to provide a better fit for real-world data compared to traditional lifetime distributions.

The survival function S (x; Φ) represents the probability that a system or component remains operational beyond a given time x . It is defined as the complement of the cumulative distribution function (CDF):

$$S(x; \Phi) = 1 - F(x; \Phi)$$

For the Kumaraswamy Marshal–Olkin Flexible Weibull Weibull Extension (KMO-FWE) distribution, the survival function is given by(Alizadeh et al., 2015b).

$$S(x;\Phi) = \left\{ 1 - \left(\frac{1-\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}{1-\bar{P}\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)} \right)^{a} \right\}^{b}$$

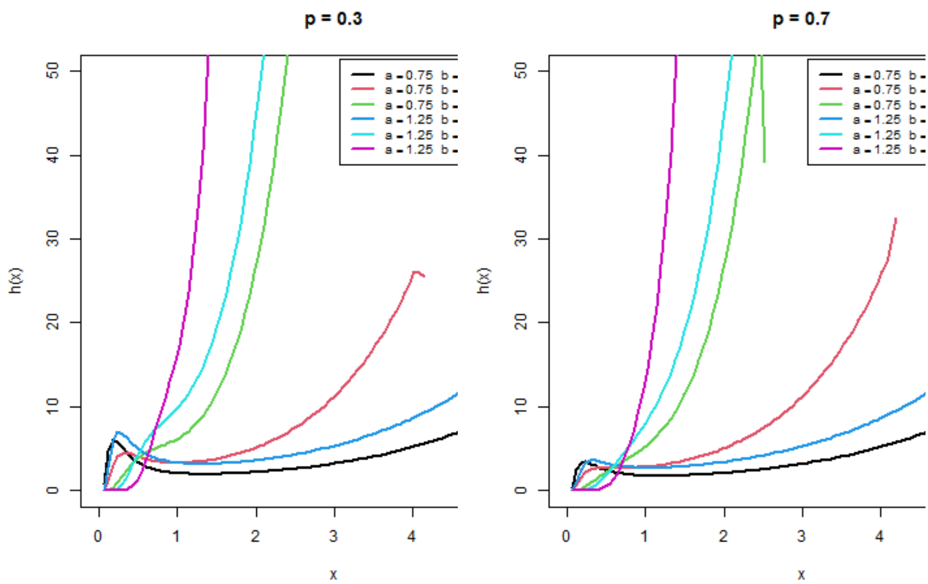
This function plays a crucial role in reliability analysis and survival studies, providing a measure of how long a subject or system is expected to function. The KMO-FWE distribution, with its enhanced flexibility, is particularly useful for modeling complex survival behaviors encountered in real-world applications.(Bebbington et al., 2007)

The hazard function $h(x;\varphi)$ also known as the failure rate function, describes the instantaneous failure rate of a system at a given time x, given that it has survived up to that time It is expressed as the quotient of the probability density function (PDF) and the survival function (SF).

$$h(x;\varphi) = \frac{f(x;\varphi)}{1-F(x;\varphi)}$$

$$h(x;\varphi) = a\,b\,(1-p)(\alpha + \frac{\beta}{x^2})\ e^{\alpha x - \frac{\beta}{x}}\ \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right) \left\{1-\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right\}^{a-1}$$

where $a, b, p, \alpha, \beta > 0$



Graph (3): Hazard function of KMOFWE distribution at different values of *p* and its parameters

This function plays a crucial role in reliability analysis and survival studies, providing a measure of how long a subject or system is expected to function. The KMO-FWE distribution, with its enhanced flexibility, is particularly useful for modeling complex survival behaviors encountered in real-world applications.

3. Expansion for the cumulative and density functions:

To facilitate the mathematical analysis and estimation of the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, the cumulative distribution function (CDF) and probability density function (PDF) can be expanded into series representations. These expansions allow for better approximations and derivation of statistical properties.

$$(1+a)^{v} = \sum_{i=0}^{\infty} \binom{v}{i} a^{i}$$

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)\, J!} z^{j}$$

$$\binom{n}{i} = \frac{n(n-1)\,...\,(n-j+1)}{i\,!}$$

$$F(x\,;\,\Phi) = 1 - \sum_{i=0}^{\infty} (-1)^{i} \binom{b}{i} \left[\frac{1 - \exp\left(-e^{\alpha x - \frac{\beta}{k}}\right)}{1 - \bar{p}\, \exp\left(-e^{\alpha x - \frac{\beta}{k}}\right)} \right]^{a\,i}$$

$$f(x;\Phi) = ab(1-P)\left(\alpha + \frac{\beta}{x^{2}}\right) \sum_{J=0}^{\infty} \sum_{t=0}^{\infty} \sum_{y=0}^{\infty} \sum_{i=0}^{\infty} \sum_{u=0}^{\infty} P^{t+u} e^{\alpha x - \frac{\beta}{x}} \left[\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right]^{j+i+u}$$
$$(-1)^{j+t+y+i+u} \binom{a-1}{j} \binom{-(a+1)}{t} \binom{b-1}{y} \binom{t+ay+1}{i} \binom{-ay+1}{u}$$

$$f(x;\,\Phi) = \sum_{j=0}^{\infty} \omega_j\, g(x;\,\Phi)$$

where

$$w_j = ab\,(1-p) \sum_{J}^{\infty} \sum_{t}^{\infty} \sum_{y}^{\infty} \sum_{i}^{\infty} \sum_{u=0}^{\infty} P^{t+u} (-1)^{j+t+u+i+u} \binom{a-1}{J}$$
$$\binom{-a-1}{t} \binom{b-1}{y} \binom{t+ay+1}{i} \binom{-ay+1}{u}$$

These expansions provide a convenient way to approximate the distribution for numerical computations and further theoretical analysis, making them useful in reliability engineering and survival analysis applications (Alizadeh et al., 2015b).

4. Properties of Kumaraswamy Marshal – Olkin Flexible Weibull Extension Distribution:

4.1. Moments:

Moments are essential statistical metrics that provide insights into the shape and characteristics of a probability distribution. For the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, the rth moment is given by:

$$\mu_r^{\backslash} = \int_0^{\infty} x^r \; g(x;\Phi)\,dx$$

Using the series expansion of the cumulative and density functions, the moments can be expressed as:

$$\mu_r^{\backslash} = \sum_{j=0}^{\infty} w_j \int_0^{\infty} x^r \; g(x;\Phi)\,dx$$

where $g(x;\Phi)$ is the probability density function (PDF) of the KMO-FWE distribution.

$$\mu_r^{\backslash} = h_{j,t,y,i,u} \int_0^{\infty} x^r \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x + \frac{\beta}{x}} \left[\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right]^{j+i+u} dx$$

$$\because \left[\exp\left(-e^{\alpha x + \frac{\beta}{x}}\right)\right]^{j+i+u} = \sum_{v=0}^{\infty} \frac{(-1)^v (j+i+u)^v}{v!} e^{v\left(\alpha x - \frac{\beta}{x}\right)}$$

$$\mu_r^{\backslash} = \omega_V \int_0^{\infty} x^r \left(\alpha + \beta x^{-2}\right) e^{(1+v)\frac{\beta}{x}} \; e^{-(1+v)\frac{\beta}{x}} \, dx$$

- where

$$\omega_r = h_{j,t,y,i,u}\ (a\ b\ p) \sum_{V=0}^{\infty} \frac{(-1)^r\ (j+i+u)^V}{v!}$$

$$\because e^{-(1+v)\frac{\beta}{x}} = \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ (u)^{-m}}{m!}$$

$$\mu_r' = \omega_v\ P^{t+u} \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ (u)^{-m}}{m!} \int_0^{\infty} x^{r-m}\ \left(\alpha + \beta x^{-2}\right) e^{(1+v)\alpha x}\ dx$$

$$\mu_r' = \omega_v\ p^{t+u} \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ (u)^{-m}}{m!} \int_0^{\infty} x^{r-m}\ \left(\alpha + \beta x^{-2}\right) e^{(1+v)\alpha x}\ dx$$

$$\mu_r' = \upsilon\left(x_{i,a,b,p,\beta}\right)\left[\frac{\alpha\ \Gamma(r-m+1)}{\alpha^{r-m+1}\ (v+1)^{r-m+1}} + \frac{\beta\ \Gamma(r-m-1)}{\alpha^{r-m-1}\ (v+1)^{r-m-1}}\right]$$

where

$$\upsilon\left(x_{i,a,b,p,\beta}\right) = \omega_v\ p^{t+u} \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ (u)^{-m}}{m!}$$

These moments provide essential insights into the central tendency, dispersion, and shape of the KMO-FWE distribution, making them useful in statistical modeling and reliability analysis (Alizadeh et al., 2015b).

4.2. Moment Generating Function:

The moment generating function (MGF) is a crucial tool in probability theory, as it provides a way to obtain all moments of a distribution. For the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, the Moment Generating Function is defined as:

$$M_x(t) = \int_0^{\infty} \exp(tx) f(x;\Phi) dx$$

- Expanding the exponential in taylor series here:-

$$\because \exp(tx) = \sum_{r=0}^{\infty} \frac{t^r}{r\,!} x^r$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r\,!} \int_0^{\infty} x^r f(x;\Phi)\, dx$$

$$M_x(t) = \gamma\left(x_{i,a,b,p,\beta}\right)\left[\frac{\alpha\,\Gamma(r-m+1)}{\alpha^{r-m+1}(v+1)^{r-m+1}} + \frac{\beta\,\Gamma(r-m+1)}{\alpha^{r-m+1}(v+1)^{r-m+1}}\right]$$

$$\gamma\left(x_{i,a,b,p,\beta}\right) = \eta_{j,t,y,i,u}\, p^{t+u} \sum_{m,r,u=0}^{\infty} \frac{(-1)^m (1+v)^m \beta^m t^r (u)^{-m}}{v\,!\; m\,!\; r\,!\; (-1)^{-v} (j+i+u)^{-v}}$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{\backslash}$$

This expansion allows for the derivation of key statistical properties, such as the mean, variance, and higher-order moments, which are useful in reliability analysis and survival studies(Alizadeh et al., 2015b).

4.3. Quantile function and simulation:

The quantile function is an essential tool in probability and statistics, as it provides a way to generate random samples from a given distribution. For the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, the quantile function is defined as the inverse of the cumulative distribution function (CDF):

$$Q(u) = F^{-1}(u)$$

$$x_u = \frac{N(u) \pm \sqrt{N(u)^2 + 4\alpha\beta}}{2\alpha}$$

$$N(u) = \log\left[\log\left(\frac{1 - p\left[1 - \left\{1 - (u)^{1/b}\right\}^{1/a}\right]}{1 - \left\{1 - (u)^{1/b}\right\}^{1/a}}\right)\right]$$

Solving for x in terms of u, the quantile function can be approximated as:

$$u = 1 - \left\{1 - \left(\frac{1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}{1 - P\exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)}\right)^a\right\}^b$$

This simulation technique is widely used in Monte Carlo methods and statistical inference, enabling researchers to evaluate the behavior of the KMO-FWE distribution under various parameter settings(Alizadeh et al., 2015b).

4.4. Skewness and kurtosis:

Skewness and kurtosis are key statistical measures that describe the shape of a probability distribution. Skewness indicates the extent to which a distribution deviates from symmetry, while kurtosis reflects the presence of heavy or light tails compared to a normal distribution. These measures help in understanding data distribution patterns and are widely used in statistical analysis. The skewness S_k of the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution is defined as:

$$S_k = \frac{Q\left(3/4\right) - 2\,Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)}$$

The kurtosis k_u is a measure of whether the distribution has heavy or light tails relative to a normal distribution. It is given by:

$$k_u = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + Q\left(1/8\right)}{Q\left(6/8\right) - Q\left(2/8\right)}$$

A normal distribution has a kurtosis of **3**. If $k_u > 3$, the distribution has heavier tails (leptokurtic), and if $k_u < 3$, it has lighter tails (platykurtic),(Alizadeh et al., 2015b).

These measures provide valuable insights into the shape of the KMO-FWE distribution, making them useful in real-world applications such as reliability analysis and risk assessment.

4.5. Order statistics:

Order statistics play a crucial role in statistical inference, particularly in reliability analysis and survival studies. For a random sample of size n drawn from the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution, the order statistics are defined as the sorted values of the sample.

The probability density function (PDF) of the rth order statistic Xr:n is given by:

$$f_{r:n}\left(x;\Phi\right) = \frac{1}{\beta\left(r, n-r+1\right)}\, f\left(x;\Phi\right)\left[F\left(x,\Phi\right)\right]^{r-1}\left[1 - F\left(x;\Phi\right)\right]^{n-r}$$

$$\because \left[1-F(x;\Phi)\right]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i} \left[F(x;\Phi)\right]^{i}$$

$$f_{r:n}(x;\Phi) = \frac{1}{\beta\,(r, n-r+1)}\, f(x;\Phi) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i} \left[F(x,\Phi)\right]^{i+r-1}$$

$$f_{r:n}(x;\Phi) = \sum_{i=0}^{n-r} \frac{(-1)^{r}\; n!}{i!\,(r-1)!(n-r-1)} \left[F(x,\Phi)\right]^{i+r-1} f(x;\Phi)$$

where:

f(x;Φ) is the PDF of the KMO-FWE distribution,

F(x;Φ) is the CDF of the KMO-FWE distribution,

n is the sample size,

r is the order of the statistic.

The flexibility of the KMO-FWE distribution makes it a strong candidate for applications requiring order statistics in real-world datasets(Alizadeh et al., 2015b).

5. Sumulation Study:

In this section, a Monte Carlo simulation is conducted to compare estimation methods using both complete data (Part I) and the Type-II censoring scheme (Part II). Monte Carlo simulation is a widely used technique in statistical analysis to evaluate the efficiency of different estimation methods under various conditions (Robert, Casella, & Casella, 1999). The estimation methods considered include Maximum Likelihood Estimation

(MLE)(Fisher, 1922), Maximum Product Spacing (MPS)(Cheng & Amin, 1983), Least Squares Estimation (LS)(Ishii, Dohi, Okamura, & Management, 2012), Weighted Least Squares Estimation (WLS)(Seber, Wild, & Sons, 2003), and Percentile Estimation (PE)(Arnold et al., 2008) The primary objective is to assess the accuracy and efficiency of these methods under different sample sizes and censoring levels. Estimating the parameters of a probability distribution is crucial for its practical application in statistical modeling. In this section, we discuss The Maximum Likelihood Estimation (MLE) method estimates parameters by maximizing the likelihood function for the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution.

$$L(\Phi) = k\log a + k\log b + k\log(1-p) + \sum_{i=1}^{k}\left[\log\left(\alpha+\frac{\beta}{t_1^2}\right)\right] + \sum_{i=1}^{k}\left[\log \upsilon(t_i)\right] + \sum_{i=1}^{k}\log\omega(t_i) + (a-1)\sum_{i=1}^{k}\log\left[1-\omega(t_i)\right]$$

$$-(a+1)\sum_{i=1}^{k}\log\{1-p\left[1-\omega(t_i)\right] + (b-1)\sum_{i=1}^{k}\log\left\{1-\left[\frac{1-\omega(t_i)}{1-P\,\omega(t_i)}\right]^{a}\right\} + (n-k)\,b\sum_{i=1}^{k}\log\left\{1-\left[\frac{1-\omega(T)}{1-P\,\omega(T)}\right]^{a}\right\}$$

$$\omega(t_i) = \exp\left(-e^{\alpha t_i - \frac{\beta}{t_i}}\right) \qquad \omega(T) = \exp\left(-e^{\alpha t_i - \frac{\beta}{T}}\right)$$

$$\upsilon(t_i) = e^{\alpha t_i - \frac{\beta}{t_i}} \qquad \upsilon(T) = e^{\alpha T - \frac{\beta}{T}}$$

$$n(t_i) = \frac{1 - \omega(t_i)}{1 - p\,\omega(t_i)} \qquad n(T) = \frac{1 - \omega(T)}{1 - p\,\omega(T)}$$

$$\frac{\partial L}{\partial a} = \frac{k}{a} + \sum_{i=1}^{k} \log\left[1 - \omega(t_i)\right] - \sum_{i=1}^{k} \log\left\{1 - P\left[1 - \omega(t_i)\right]\right\}$$
$$- (b-1)\sum_{i=1}^{k} \frac{n(t_i)^a \log n(t_i)}{1 - n(t_i)^a} - (n-k)\, b \sum_{i=1}^{k} \frac{n(T)^a \log n(t)}{1 - n(T)^a}$$

$$\frac{\partial L}{\partial b} = \frac{k}{b} + \sum_{i=1}^{k} \log\left(1 - n(t_i)^a\right) + (n-k)\sum_{i=1}^{k} \log\left(1 - n(T)^a\right)$$

$$\frac{\partial L}{\partial p} = \frac{-k}{1-P} + (a+1) \sum_{i=1}^{k} \frac{1 - \omega(t_i)}{1 - p\left[1 - \omega(t_i)\right]} + (b-1)\sum_{i=1}^{k} \frac{-a\, n(t_i)^{a-1}\, \omega(t_i)\left[1 - \omega(t_i)\right]}{\left[1 - n(t_i)^a\right]\left[1 - P\,\omega(t_i)\right]^2}$$
$$+ (n-k)\, b \sum_{i=1}^{k} \frac{-a\, n(T)^{a-1}\, \omega(T)\left[1 - \omega(T)\right]}{\left[1 - n(T)^a\right]\left[1 - p\,\omega(T)\right]^2}$$

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{k} \frac{1}{\alpha + \frac{\beta}{t_i^2}} + \sum_{i=1}^{k} t_i - \sum_{i=1}^{k} t_i\, \upsilon(t_i) + (a-1) \sum_{i=1}^{k} \frac{t_i\, \omega(t_i)\, \upsilon(t_i)}{1 - \omega(t_i)}$$

$$+ \; p(a+1)\sum_{i=1}^{k}\frac{t_i\,\omega(t_i)\,\upsilon(t_i)}{1-p\,[1-\omega(t_i)]} - a(b-1)\sum_{i=1}^{k}\frac{n(t_i)^{a-1}\,\frac{\partial n(t_i)}{\partial \alpha}}{1-n(t_i)^{a}}$$

$$+ \; b(n-k)\sum_{i=1}^{k}\frac{n(T)^{a-1}\,\frac{\partial n(T)}{\partial \alpha}}{1-n(T)^{a}}$$

$$\frac{\partial n(t_i)}{\partial \alpha} = \frac{[1-p\,\omega(t_i)]\,[t_i\,\omega(t_i)\,\upsilon(t_i)] - [1-\omega(t_i)][(p)(t_i)\omega(t_i)\upsilon(t_i)]}{[1-p\,\omega(t_i)]^2}$$

$$\frac{\partial n(T)}{\partial \alpha} = \frac{[1-p\,\omega(T)][T\,\omega(T)\,\upsilon(T)] - [1-\omega(T)]\,[(p)(T)\omega(T)\upsilon(T)]}{[1-p\,\omega(T)]^2}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{k}\frac{1/t_i^2}{\alpha+\frac{\beta}{t_i^2}} + \sum_{i=1}^{k}\frac{1}{t_i} - \sum_{i=1}^{k}\frac{1}{t_i}\,\upsilon(t_i) - (a-1)\sum_{i=1}^{k}\frac{1}{t_i}\,\frac{\omega(t_i)\upsilon(t_i)}{[1-\omega(t_i)]}$$

$$-p(a+1)\sum_{i=1}^{k}\frac{1}{t_i}\,\frac{\omega(t_i)\upsilon(t_i)}{1-P[1-\omega(t_i)]} - a(b-1)\sum_{i=1}^{k}\frac{n(t_i)^{a-1}\,\frac{\partial n(t_i)}{\partial \beta}}{1-n(t_i)^{a}}$$

$$+ \; b\,(n-k)\sum_{i=1}^{k}\frac{n(T)^{a-1}\,\frac{\partial n(T)}{\partial \beta}}{1-n(T)^{a}}$$

$$\frac{\partial n(t_i)}{\partial \beta} = \frac{[1-p\,\omega(t_i)]\left[\frac{1}{(t_i)}\,\omega(t_i)\upsilon(t_i)\right] + [1-\omega(t_i)]\left[(p)\frac{1}{t_i}\,w(t_i)\upsilon(t_i)\right]}{[1-p\,\omega(t_i)]^2}$$

$$\frac{\partial n(T)}{\partial \beta} = \frac{[1 - p\,\omega(T)]\left[\frac{1}{(t_i)}\omega(T)\upsilon(T)\right] + [1-\omega(T)]\left[(p)\frac{1}{t_i}w(T)\upsilon(T)\right]}{[1 - p\,\omega(T)]^2}$$

The Maximum Likelihood Estimation (MLE) method remains one of the most widely used approaches for parameter estimation due to its desirable statistical properties, such as asymptotic efficiency and consistency. However, for complex distributions like the KMO-FWE distribution, solving the likelihood equations analytically is often challenging. As a result, numerical optimization techniques . While MLE generally provides accurate estimates, it may suffer from convergence issues or bias in small samples, necessitating the use of alternative estimation techniques or improved initialization strategies for better performance.

5.1. Estimation under complete sample:

For the complete sample case, the simulation follows these steps:

A total of 1,000 replications are performed for each scenario(Mooney, 1997),Number of replications = 1000.
Sample sizes are: n=40,60,100.(Mudholkar, Srivastava, & Kollia, 1996)
The Average (AVG) and Root Mean Square Error (RMSE)(Hyndman & Koehler, 2006) are computed for each estimation method to evaluate their performance.
Methods of estimation are: MLE, MPS, LSE, WLSE , PE
Parameters of the Kumaraswamy Marshal–Olkin Flexible Weibull Extension distribution are:
$a=0.75, b=1.5, \alpha=0.5, \beta=0.5$
$a=0.75, b=1.5, \alpha=1.5, \beta=1.5$
$a=1.50, b=2.5, \alpha=2.5, \beta=2.5$
Computed measure: Average (Avg.) and root mean square error (RMSE)

Part I: Complete Case

Table (1.a): Average and RMSE for different estimation methods of KMOFWE distribution at different sample sizes and $\boldsymbol{a = 0.75, b = 1.5, \alpha = 0.5, \beta = 0.5}$

n	Parm	Method of Estimation									
		MLE		MPS		LSE		WLSE		PE	
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE
$\boldsymbol{p = 0.3}$											
40	a	1.6628	1.7522	2.3409	2.3017	0.8843	0.7532	0.8052	0.5782	0.9281	0.5573
	b	2.0517	1.3657	2.4971	1.9460	1.8478	1.2035	1.8520	0.9210	1.6639	0.6096
	α	0.7297	0.6622	0.9332	1.0226	0.6003	1.2776	0.3481	0.7353	0.4996	0.6301
	β	0.4822	0.3191	0.4409	0.4469	0.6228	0.3640	0.6372	0.4008	0.5331	0.4237
60	a	1.4914	1.5833	2.2139	2.2631	0.6978	0.4006	0.7454	0.3840	0.9156	0.5701
	b	1.9374	1.1693	2.3976	1.7435	1.8031	0.8164	1.6790	0.5420	1.6262	0.5401
	α	0.6560	0.5113	0.7578	0.6598	0.3966	0.7866	0.3876	0.5072	0.4779	0.4678
	β	0.4760	0.2556	0.4124	0.3211	0.6346	0.2742	0.5961	0.2499	0.5312	0.3473
100	a	1.1211	1.0561	1.6780	1.7201	0.7413	0.3161	0.7794	0.4030	0.9469	0.5514
	b	1.7161	0.6981	1.9925	1.0823	1.6040	0.5152	1.6304	0.4016	1.5991	0.4149
	α	0.5899	0.3087	0.6738	0.3956	0.5186	0.7256	0.4318	0.3974	0.4612	0.3122
	β	0.5152	0.2371	0.4357	0.2470	0.5667	0.2045	0.5619	0.2036	0.4993	0.2450
$\boldsymbol{p = 0.7}$											
40	a	1.2843	1.3282	2.2343	2.1507	0.8238	0.5642	0.8816	0.6768	0.8610	0.4369
	b	2.2072	2.0169	3.5735	3.5803	1.7202	1.1678	1.6387	0.8273	1.6240	0.7063
	α	0.6195	0.3698	0.6016	0.4069	0.6026	0.8067	0.5779	0.5039	0.5318	0.4302
	β	0.5972	0.4535	0.4203	0.3904	0.6909	0.5517	0.6093	0.4197	0.5436	0.3884
60	a	1.0589	1.0237	1.8123	1.8113	0.9947	0.7449	0.8263	0.4550	0.8468	0.3867
	b	1.9222	1.4090	2.9154	2.7637	1.8369	1.1311	1.6681	0.5565	1.5970	0.5834
	α	0.5542	0.2515	0.5527	0.2851	0.4742	0.4065	0.4519	0.2581	0.4738	0.2610

	β	0.5928	0.3624	0.4953	0.4190	0.5098	0.2385	0.5513	0.2233	0.5571	0.4138
100	a	1.0207	0.8359	1.5800	1.5108	0.8351	0.4175	0.8267	0.4639	0.8866	0.4303
	b	1.8134	1.0311	2.4530	2.0283	1.6849	0.5107	1.7209	0.7801	1.6227	0.5331
	α	0.5160	0.1710	0.5149	0.1822	0.4639	0.3592	0.4344	0.2480	0.4722	0.1781
	β	0.5465	0.3054	0.4610	0.3229	0.5380	0.2311	0.5518	0.2302	0.5081	0.3355

Table (1.b): Average and RMSE for different estimation methods of KMOFWE distribution at different sample sizes and $\boldsymbol{a = 0.75, b = 1.5, \alpha = 1.5, \beta = 1.5}$

n	Parm	Method of Estimation									
		MLE		**MPS**		**LSE**		**WLSE**		**PE**	
		AVG	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**
$\boldsymbol{p = 0.3}$											
40	a	1.3273	1.3436	1.8926	1.8462	1.0097	1.0208	1.1245	1.2974	0.9455	0.7997
	b	1.7329	2.0185	1.6805	1.8862	2.1132	2.1348	2.3694	3.0395	2.2849	1.9368
	α	2.0026	1.0901	2.3068	1.5246	1.7584	1.2858	1.7888	1.2585	1.3166	0.9683
	β	1.4930	0.8348	1.3984	1.3001	1.7401	0.8461	1.7493	0.8899	1.6710	0.7313
60	a	1.2116	1.1293	1.8379	1.8461	0.9599	1.0593	0.9535	1.0594	0.9097	0.8570
	b	1.5839	1.2040	1.6019	1.5788	2.1064	2.0780	2.2537	2.2313	2.1169	1.7382
	α	1.8902	1.0273	2.0667	1.1318	1.7869	1.2444	1.6620	1.3303	1.4670	0.9740
	β	1.4767	0.7584	1.2453	0.9188	1.7793	0.7920	1.8177	0.8753	1.7610	0.8666
100	a	1.1091	1.0298	1.5836	1.5449	0.8430	0.8062	0.7311	0.5211	0.8417	0.5457
	b	1.6809	1.1364	1.6209	1.3104	2.0687	2.1838	1.9186	1.2193	2.3208	1.7089
	α	1.6676	0.6763	1.8061	0.7744	1.6905	0.9439	1.5371	0.7294	1.2411	0.8379
	β	1.4842	0.6134	1.2854	0.7770	1.7932	0.7256	1.8197	0.6000	1.7785	0.7683
$\boldsymbol{p = 0.7}$											
40	a	1.0154	0.9301	1.6485	1.5739	0.7616	0.5713	0.7898	0.7284	0.8056	0.6759

	b	1.7507	2.3778	2.3548	3.5927	2.0314	2.8573	2.2214	3.0691	2.3508	2.7849
	α	1.8954	0.9168	1.9215	1.0671	1.7367	1.0355	1.7320	0.9789	1.6498	1.0208
	β	1.5892	0.8874	1.3447	1.2023	1.7550	0.8992	1.8213	0.9532	1.8264	0.9382
60	a	0.8996	0.7211	1.4853	1.3720	0.7961	0.6016	0.7410	0.5319	0.7765	0.5377
	b	1.4932	1.1237	1.9132	2.2440	2.1537	2.2259	1.8930	2.2630	2.2375	2.3353
	α	1.8437	0.7434	1.7896	0.7948	1.7601	1.0923	1.7307	0.7963	1.5276	0.7026
	β	1.6869	0.9523	1.3468	1.0289	1.7524	0.8401	1.7676	0.6937	1.7862	0.7768
100	a	0.9545	0.7915	1.2779	1.1215	0.8510	0.5800	0.7832	0.5148	0.7983	0.4296
	b	1.7603	1.8098	1.8525	2.1072	2.1024	2.0306	1.8867	1.7624	1.8147	1.2180
	α	1.6587	0.5157	1.7072	0.6450	1.6351	0.7949	1.6556	0.6878	1.5420	0.5462
	β	1.5778	0.6459	1.4140	0.8598	1.6498	0.6046	1.6982	0.5659	1.6065	0.4642

Table (1.c): Average and RMSE for different estimation methods of KMOFWE distribution at different sample sizes and $a = 1.25, b = 2.5, \alpha = 2.5, \beta = 2.5$

n	Parm	Method of Estimation									
		MLE		**MPS**		**LSE**		**WLSE**		**PE**	
		AVG	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**	**AVG**	**RMSE**
$p = 0.3$											
40	a	2.2750	2.2726	2.5251	2.6409	1.7823	1.5134	1.7174	1.4475	1.6056	1.3003
	b	2.8713	3.8895	3.0588	4.7204	3.2598	3.1005	3.4117	3.0498	2.9295	1.9874
	α	3.5866	2.0634	4.2415	3.1366	3.4392	2.6557	3.1032	1.9375	2.9570	1.7791
	β	2.8580	2.0016	3.2456	2.9864	3.0528	1.8760	2.9725	1.6226	2.8567	1.4432
60	a	2.0734	2.0649	2.5089	2.4629	1.5475	1.1340	1.9877	1.6008	1.4801	1.0312
	b	2.9605	3.7945	3.0777	4.7530	3.3545	2.5345	3.5377	2.4666	3.3343	2.4641
	α	3.3499	1.6652	3.6841	1.9021	2.8858	1.5451	2.7066	1.6390	2.5681	1.1404
	β	2.7709	1.8255	2.7939	2.1077	2.8768	1.3654	2.5951	1.3255	2.7303	1.1079

100	a	1.7923	1.5449	2.3172	2.2175	1.6436	1.2158	1.5221	1.0754	1.5953	1.2075
	b	2.5782	3.0868	2.7314	3.8104	3.0898	2.2731	3.0477	2.2371	3.1034	1.8341
	α	3.2043	1.3420	3.4555	1.6117	2.8471	1.2975	2.8736	1.3024	2.6308	1.0971
	β	2.6183	1.2100	2.6398	1.7574	2.7188	1.1389	2.8178	1.2117	2.6626	1.0959
$p = 0.7$											
40	a	2.2878	2.6178	2.4490	2.6256	1.7839	1.6158	1.9282	1.6767	1.7075	1.4868
	b	4.0523	5.8917	3.2968	3.9385	3.6201	3.7476	3.5348	3.1221	3.3584	2.8110
	α	3.1854	1.7883	3.7501	2.6063	3.1202	1.8589	2.8010	1.5692	2.8106	1.6433
	β	2.7804	1.8409	3.1601	2.9120	2.8878	1.6110	2.6723	1.5414	2.7132	1.5031
60	a	1.7100	2.0717	2.0772	2.2747	1.2906	1.0714	1.4825	1.2340	1.4553	1.1743
	b	3.1666	4.0894	3.5126	4.5888	2.9822	2.5750	3.2419	2.6364	3.1273	2.5681
	α	3.4110	1.8356	3.6404	2.0742	3.3182	1.9380	3.2054	1.8176	2.9492	1.5891
	β	3.0583	1.8386	3.2977	2.6265	3.2506	1.6919	3.0567	1.5733	2.9118	1.5093
100	a	1.7820	1.6185	2.0478	2.0459	1.4565	1.1043	1.8346	1.3803	1.3600	0.9690
	b	3.7334	4.2969	3.0242	3.4654	3.1787	2.5719	3.9675	3.4342	3.0967	2.1185
	α	2.8736	1.2459	3.2587	1.5450	2.9560	1.5129	2.5364	1.1960	2.8045	1.1681
	β	2.7179	1.5578	2.8138	1.9085	2.8713	1.3691	2.4797	1.1613	2.8445	1.2210

The simulation results provide insights into the effectiveness of each estimation method under different data conditions. MLE is known for its asymptotic efficiency, but it may suffer from bias or convergence issues in small samples(Lehmann & Casella, 2006). MPS serves as a stable alternative, particularly when MLE struggles with boundary estimates(Cheng & Amin, 1983).

5.2. Estimation under Type-II censoring:

Type-II censoring is a widely used censoring scheme in survival analysis and reliability studies, where a fixed number rrr of failures is observed out of a total sample size n, and the remaining observations are censored. This approach is particularly relevant in life-testing experiments where the study continues until a predetermined number of failures occur, making it an effective method for analyzing lifetime data (Lawless, 2011),In this study, we evaluate the performance of different parameter estimation methods for the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution under Type-II censoring. The estimation process involves:

- Generating random samples of size n=60,100,150 from the KMO-FWE distribution.
- Applying Type-II censoring, where the number of observed failures is determined by r=f*n

where:

f=40% ,60% ,80% and n=60,100,150 represents different censoring levels(Nelson, 2005).

- Number of replications = 1000
- Assessing the accuracy of the estimators using statistical metrics such as Average Estimates (AVG) and Root Mean Square Error (RMSE).

Part II: Censoring case (Type-II)

Table (2.a): Average and RMSE for different estimation methods of KMOFWE distribution under Type-II censoring at different sample sizes and $a = 0.75, b = 1.5, \alpha = 0.5, \beta = 0.5$

p	n	Parm	Number of failure (r)					
			$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
			AVG	RMSE	AVG	RMSE	AVG	RMSE
0.3	60	a	2.0970	2.7674	1.1966	1.6701	1.3198	1.5099
		b	3.6868	5.2849	2.6080	3.6955	2.0924	1.9835
		α	3.3650	4.0548	3.2822	3.8612	1.7220	1.9970
		β	0.6975	0.6117	0.7799	0.6367	0.5928	0.4404
	100	a	1.5243	2.0368	1.1731	1.5301	1.1285	1.2221
		b	3.0893	4.2551	2.4724	3.2635	1.8928	1.3520
		α	3.0983	3.8054	3.0607	3.5992	1.6081	1.7081
		β	0.6626	0.5295	0.6600	0.4634	0.5648	0.3338
	150	a	1.3318	1.5420	0.9773	1.1032	1.0536	1.0586
		b	2.3882	3.0303	1.9732	1.8207	1.7755	0.9228
		α	3.5137	4.1552	2.7446	3.1855	1.3556	1.3361
		β	0.6152	0.4560	0.6078	0.3616	0.5523	0.2890
0.7	60	a	1.4275	1.7664	1.1988	1.5167	1.0653	1.1723
		b	4.6910	6.7090	3.1224	4.3952	2.0395	2.0636
		α	3.4637	4.1648	2.0067	2.4441	1.0568	1.1270
		β	0.7947	0.8149	0.7542	0.6926	0.6198	0.4008
	100	a	1.0487	1.3138	1.0439	1.1963	0.9469	0.8551
		b	3.9115	6.0451	2.4467	3.3928	1.8237	1.2830

		α	3.2544	3.9204	1.7686	1.9574	0.7827	0.6605
		β	0.7925	0.6734	0.6641	0.5080	0.5916	0.3194
	150	a	0.8390	0.7831	1.0149	1.1306	0.8980	0.7651
		b	3.0245	4.7243	2.1047	2.1138	1.6945	0.9637
		α	3.0811	3.7874	1.3712	1.4299	0.6864	0.5121
		β	0.6945	0.4834	0.6096	0.3502	0.5892	0.3116

Table (2.b): Average and RMSE for different estimation methods of KMOFWE distribution under Type-II censoring at different sample sizes and $a = 0.75, b = 1.5, \alpha = 1.5, \beta = 1.5$

p	n	Parm	Number of failure (r)					
			$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
			AVG	RMSE	AVG	RMSE	AVG	RMSE
0.3	60	a	1.7833	2.4672	1.4178	2.1336	1.2675	1.6967
		b	3.1249	4.0662	2.3140	3.9464	1.8769	2.2956
		α	5.3476	7.2572	3.9882	4.1129	2.7089	2.1585
		β	2.2358	1.8835	2.0456	1.5962	1.7342	1.1182
	100	a	1.6244	2.1407	1.2226	1.7903	0.9152	1.0779
		b	3.2231	4.5231	2.0722	2.8469	1.7336	1.9379
		α	4.9689	6.2473	3.3986	3.1276	2.3632	1.5530
		β	1.9539	1.5521	1.9078	1.2955	1.7325	0.8615
	150	a	1.2828	1.5559	1.0165	1.1810	0.9228	0.9720
		b	3.1329	4.2398	2.0428	2.6668	1.7845	1.9837
		α	4.1502	4.5018	2.8663	2.5602	2.2220	1.4136
		β	1.7937	1.1966	1.7377	1.0147	1.6674	0.8257
0.7	60	a	1.3998	1.6736	1.3577	1.6773	1.0446	1.1739

		b	3.8907	6.1524	3.0456	4.9402	2.0421	3.1732
		α	3.9389	4.8012	2.6419	2.3854	2.1473	1.2986
		β	1.9334	1.8279	1.8138	1.6108	1.5749	0.8490
	100	a	1.3294	1.4260	1.2149	1.3003	1.0574	1.1251
		b	3.5667	5.6596	2.6610	3.8234	2.2013	3.2445
		α	3.1411	3.1618	2.0944	1.5644	1.8768	1.0186
		β	1.6785	1.4173	1.5294	0.9822	1.5928	0.8330
	150	a	1.0691	1.0260	1.0228	1.0784	0.9460	0.8775
		b	3.1372	4.9911	2.5033	3.7790	1.9452	2.1386
		α	2.8544	2.9286	2.0722	1.5359	1.7162	0.8060
		β	1.7262	1.2458	1.6349	0.9754	1.5627	0.6632

Table (2.c): Average and RMSE for different estimation methods of KMOFWE distribution under Type-II censoring at different sample sizes and $a = 1.25, b = 2.5, \alpha = 2.5, \beta = 2.5$

p	n	Parm	Number of failure (r)					
			$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
			AVG	RMSE	AVG	RMSE	AVG	RMSE
0.3	60	a	3.0298	3.0658	2.8688	2.8977	2.6192	2.8861
		b	5.7440	8.8032	3.9260	5.3737	4.3352	5.7855
		α	4.7555	5.4834	4.3793	4.1064	3.8737	2.9169
		β	2.9496	2.4887	3.0724	2.5647	3.2088	2.5307
	100	a	2.7491	2.9453	2.4337	2.6459	2.4122	2.4865
		b	4.8074	7.2266	4.4686	5.7477	4.2782	5.0116
		α	4.2681	4.3342	4.2199	3.8200	3.2726	2.1350

		β	2.9098	2.2775	3.1558	2.4189	2.9270	2.1517
	150	a	2.4805	2.3688	2.7060	2.5242	2.0161	1.7953
		b	4.0384	5.2519	4.6380	5.3862	3.5902	4.4869
		α	4.2051	4.4485	3.0886	2.3665	3.1625	1.7844
		β	2.8396	2.2106	2.5085	1.8474	2.7454	1.7186
0.7	60	a	2.7715	2.8376	2.5695	2.5302	2.3295	2.3558
		b	5.8670	7.6114	4.6003	5.4123	4.6315	5.9333
		α	4.3745	4.9211	3.9721	4.1991	3.4604	2.7828
		β	3.0907	2.9021	3.1837	3.1431	2.9803	2.4290
	100	a	2.5049	2.6092	2.4059	2.1960	1.8416	1.8064
		b	5.0410	6.6706	5.6819	6.7502	3.6828	4.1748
		α	4.1824	4.1548	2.7742	2.0793	3.2854	1.9324
		β	2.9490	2.4106	2.5739	1.8375	2.9649	1.8663
	150	a	2.2071	2.0761	2.3968	2.1208	2.0397	1.8220
		b	4.9198	5.9043	5.0827	6.0198	4.3903	4.9905
		α	3.6100	3.2426	2.8243	2.1741	2.7664	1.5036
		β	2.7859	1.9515	2.5428	1.9432	2.6221	1.5638

Comments:

1. From the results obtained in Tables 1.a, 1.b, and 1.c for the complete sampling, we can observe the following:
   a) As the sample size (n) increases, the RMSE decreases, and the AVG value approaches the true initial value for all KMOFW distribution parameters, namely: $a, b, \alpha,$ and β.
   b) As the value of p increases, we observe a decrease in RMSE.
   c) As the values of parameters a and b increase, the RMSE increases.

   d) Comparing the estimation methods, we find that the PE method has the lowest RMSE.
2. From the results obtained in Tables 2.a, 2.b, and 2.c for the Type-II censoring case, in addition to the previous observations, we note that as the failure proportion (f) increases, the RMSE decreases, while all other variables remain constant.

Conclusion:

In this study, the Kumaraswamy Marshal–Olkin Flexible Weibull Extension (KMO-FWE) distribution was introduced and analyzed as a flexible model capable of capturing various hazard rate behaviors. Several key statistical properties were derived, including the cumulative distribution function (CDF), probability density function (PDF), survival function, hazard function, moments, moment generating function, quantile function, and order statistics. Furthermore, multiple parameter estimation methods were explored, with a particular focus on Maximum Likelihood Estimation (MLE).To evaluate the performance of these estimation methods, a Monte Carlo simulation study was conducted using both complete data and Type-II censored data. The simulation results provided valuable insights into the efficiency and accuracy of different estimation techniques:

MLE demonstrated strong performance in large samples but showed potential bias and convergence issues in small samples.

Maximum Product Spacing (MPS) provided stable estimates, especially when MLE struggled with boundary values.

Least Squares (LS) and Weighted Least Squares (WLS) were computationally simpler but less robust under censoring.

Percentile Estimation (PE) proved useful for scenarios where percentile-based estimates are preferable, particularly in reliability analysis.

The results indicate that the KMO-FWE distribution is a powerful and flexible model for analyzing lifetime data, survival studies, and reliability engineering applications. The choice of an appropriate estimation method depends on factors such as sample size, censoring level, and computational feasibility.

For future research, further extensions of the KMO-FWE distribution could be explored, including Bayesian estimation methods, regression modeling, and applications to real-world datasets in engineering and biomedical sciences.

الملخص العربي:

تقدم هذه الدراسة إمتداد لتوذيع ويبل المرن كوماراسوامي مارشال–أولكين(KMO–FWE) ، وهو نموذج احتمالي تم تطويره حديثًا كامتداد لتوزيعات العمر الحالية. يعزز هذا الامتداد القدرة على نمذجة أنماط مختلفة لمعدلات المخاطر بمرونة أكبر. تستكشف الدراسة الخصائص الأساسية لهذا التوزيع، بما في ذلك دالة التوزيع التراكمي(CDF) ، ودالة كثافة الاحتمال (PDF)، ودالة الصلاحية، ودالة المخاطر، والعزوم، ودالة توليد العزوم، والإحصاءات الترتيبية. لتقدير معلمات التوزيع، يتم تقييم العديد من الأساليب الإحصائية، مع التركيز الأساسي على تقدير الإمكان الأكبر (MLE).بالإضافة إلى ذلك، يتم النظر في تقنيات تقدير بديلة مثل وطريقة (MPS)، والمربعات الصغرى(LS) ، والمربعات الصغرى المرجحة (WLS)، وتقدير النسبة المئوية .(PE) يتم إجراء دراسة محاكاة مونت كارلو لتقييم دقة وكفاءة أساليب التقدير هذه في ظل كل من البيانات الكاملة وظروف

الرقابة من النوع الثاني. تشير النتائج إلى أن تقدير MLE فعال للغاية في العينات الكبيرة، ولكنه قد يُحدث تحيزًا في العينات الأصغر. في المقابل، يوفر MPS تقديرات مستقرة للمعلمات في ظل ظروف أكثر تعقيدًا. في حين أن LS و WLSسهلتا الحساب، إلا أن أدائهما يتراجع في وجود الرقابة. في الوقت نفسه، يُثبت PE فائدته في تحليل الموثوقية القائم على تقدير النسبة المئوية. بشكل عام، تُبرز النتائج توزيع KMO-FWE كنموذج متعدد الاستخدامات وموثوق، ومناسب للتطبيقات في هندسة الموثوقية، وتحليل الصلاحية، والبحوث الطبية الحيوية.

Reference

1- Ahmad, H. H., Almetwally, E. J. P. J. o. S., & Research, O. (2020). Marshall-Olkin generalized Pareto distribution: Bayesian and non Bayesian estimation. 21-33.
2- Alizadeh, M., Tahir, M., Cordeiro, G. M., Mansoor, M., Zubair, M., & Hamedani, G. J. J. o. t. E. M. S. (2015a). The Kumaraswamy marshal-Olkin family of distributions. *23*(3), 546-557.
3- Alizadeh, M., Tahir, M., Cordeiro, G. M., Mansoor, M., Zubair, M., & Hamedani, G. J. J. o. t. E. M. S. (2015b). The Kumaraswamy marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society, 23*(3), 546-557.
4- Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (2008). *A first course in order statistics*: SIAM.
5- Bebbington, M., Lai, C.-D., Zitikis, R. J. R. E., & Safety, S. (2007). A flexible Weibull extension. *92*(6), 719-726.
6- Cheng, R., & Amin, N. J. J. o. t. R. S. S. S. B. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *45*(3), 394-403.
7- Cordeiro, G. M., De Castro, M. J. J. o. s. c., & simulation. (2009). A new family of generalized distributions. *Journal of Statistical Computation and Simulation, 81*(7), 883-898.
8- Fisher, R. A. J. P. t. o. t. R. S. o. L. S. A., containing papers of a mathematical or physical character. (1922). On the mathematical foundations of theoretical statistics. *222*(594-604), 309-368.
9- Hyndman, R. J., & Koehler, A. B. J. I. j. o. f. (2006). Another look at measures of forecast accuracy. *22*(4), 679-688.
10- Ishii, T., Dohi, T., Okamura, H. J. Q. T., & Management, Q. (2012). Software reliability prediction based on least squares estimation. *9*(3), 243-264.
11- Kuhl, M. E., Wilson, J. R. J. J. o. S. C., & Simulation. (2000). Least squares estimation of nonhomogeneous Poisson processes. *67*(1), 699-712.

12- Kumaraswamy, P. J. J. o. h. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology, 46*(1-2), 79-88.
13- Lawless, J. F. (2011). *Statistical models and methods for lifetime data*: John Wiley & Sons.
14- Lehmann, E. L., & Casella, G. (2006). *Theory of point estimation*: Springer Science & Business Media.
15- Marshall, A. W., & Olkin, I. J. B. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *84*(3), 641-652.
16- Mooney, C. Z. (1997). *Monte carlo simulation*: Sage.
17- Mudholkar, G. S., Hutson, A. D. J. C. i. S.-.-T., & Methods. (1996). The exponentiated Weibull family: some properties and a flood data application. *25*(12), 3059-3083.
18- Mudholkar, G. S., Srivastava, D. K., & Kollia, G. D. J. J. o. t. A. S. A. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *91*(436), 1575-1583.
19- Nelson, W. B. (2005). *Applied life data analysis*: John Wiley & Sons.
20- Robert, C. P., Casella, G., & Casella, G. (1999). *Monte Carlo statistical methods* (Vol. 2): Springer.
21- Seber, G. A., Wild, C. J. J. N. J. J. W., & Sons. (2003). Nonlinear regression. hoboken. *62*(63), 1238.

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